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Solutions of a class of nonlinear master equations

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Nonlinear master equations in the case of one kind of particle are discussed from the point of view of solving a martingale problem. We get some conditions from which the existence, uniqueness and ergodicity of solutions follow. These results are then used to discuss the phenomenon of phase transitions of the second Schlögl model.

nonlinear master equation * pure jump Markov process * martingale * birth-death process * invariant measure

1. Introduction

Nonlinear master equations were first proposed by Nicolis and Prigogine [12] and later Yan and Li reestablished them using probability methods [15]. The equations describe a class of nonlinear pure jump Markov processes in chemistry, physics and biology. There exist analogues of nonlinear diffusion processes for which the qualitative theory was studied extensively by Dawson [3], Dawson and Gärtner [4], Funaki [7, 8] and many others. In [13], Shiga and Tanaka studied some qualitative properties for bounded nonlinear master equations. As for the unbounded case, the theory has eluded clarification. The objective of this paper is to study the qualitative properties of a class of nonlinear master equations with unbounded jump rate.

Here is some of the notation to be used in this paper. Let $E=\{0,1,2,\ldots\}$ be equipped with the discrete topology. $D([0,\infty),E)$ denotes the space of functions from $[0,\infty)$ to E that are right-continuous and have left-hand limits with the Skorohod metric d on it. It is well known that $(D([0,\infty),E),d)$ is a complete separable space in which the Borel σ-algebra coincides with $\mathscr{F}=\sigma\{X_t\colon t\geq 0\}$, the

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smallest σ-field generated by $\{X_t: t \geqslant 0\}$ where $X_t(\omega) = \omega(t)$ for all $\omega \in D([0, \infty), E)$. For each $t \geqslant 0$, let $\mathscr{F}_t = \sigma\{X_s: 0 \leqslant s \leqslant t\}$ be the smallest σ-field generated by $\{X_s: 0 \leqslant s \leqslant t\}$. $\mathscr{P}(D([0, \infty), E), \mathscr{F})$ is the set of all probability measures on $(D([0, \infty), E), \mathscr{F})$. $\mathscr{P}_p(E)$ is the set of all probability measures on E with finite p order moment for $p > 0$. For any non-negative function $g(t)$, define the following operator:

$$\Omega_{g(t)} f(i) = \sum_{j \in E} q_{i,j}(f(j) - f(i)) + g(t)(f(i+1) - f(i)),$$

$i \in E, f \in B(E)$, where $B(E)$ is the set of all bounded functions on E, $Q = (q_{i,j})_{i,j} \in E$ is a conservative totally stable Q-matrix satisfying the following conditions:

$$\forall k \in E, \quad \sup_{\substack{i \in E \\ i \neq k}} q_{i,k} < \infty. \tag{1.1}$$

There exists $p > 1$, $M > 0$, $\lambda > 1$ such that

$$\sum_{j \in E} q_{i,j}(j^q - i^q) \leqslant M + \lambda i^q, \quad i \in E, \ q = 1 \text{ or } p. \tag{1.2}$$

$$C = \sup\{g(j_1, j_2): j_2 > j_1 \geqslant 0, j_1, j_2 \in E\} < \infty, \tag{1.3}$$

where

$$\begin{aligned} g(j_1, j_2) = \sum_{k \neq 0} & (q_{j_2, j_2+k} - q_{j_1, j_1+k}) k (j_2 - j_1)^{-1} \\ & + 2 \sum_{k=1}^{\infty} [(q_{j_2, 1-k} - q_{j_1, 2j_1 - j_2 + k})^+ \\ & \qquad + (q_{j_1, j_2+k} - q_{j_2, 2j_2 - j_1 + k})^+] \cdot k (j_2 - j_1)^{-1} \end{aligned}$$

and $q_{i,j} = 0$ if $j < 0$.

These conditions are satisfied by many models with one kind of particle such as the Schlögl first model, the Schlögl second model, the linear growth model and the autocatalytic reaction model [12]. The corresponding Kolmogorov forward equations of these models are referred to as linear master equations which describe the chemical reaction processes in a container. Nicolis and Prigogine also considered the systems of chemical reactions with interactions between the inside and outside of a container. The systems can be described by the solutions of non-linear master equations as follows.

Definition 1.1. Let $u \in \mathscr{P}_1(E)$. $P \in \mathscr{P}(D, \mathscr{F})$ is called a solution of a nonlinear master equation with initial distribution u if its marginal distribution $u_t(\cdot) = P \circ X_t^{-1}(\cdot)$ satisfies the nonlinear master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t(x) = \sum_{y \in E} u_t(y) \Omega_{\|u_t\|} I_{\{x\}}(y), \quad u_0(x) = u(\{x\}), \tag{1.4}$$

where

$$u_t(x) = P\{X_t = x\}, \qquad \|u_t\| = \sum_{y\in E} u_t(y)y.$$

$P \in \mathscr{P}(D, \mathscr{F})$ is called a q-solution of (1.4) if, in addition, for every $j \in E$.

$$P\{X_{t+s} = j \mid \mathscr{F}_t\}(\cdot) = p(t, X_t;\ t+s, j), \quad P\text{-a.s.}, \tag{1.5}$$

where $p(s, i;\ t, j)$ is a transition function [6] satisfying

$$\frac{\mathrm{d}}{\mathrm{d}s} p(t, i;\ t+s, j) = \sum_k p(t, i;\ t+s, k)(\Omega_{\|u_{t+s}\|} I_{\{j\}})(k), \quad t \geqslant 0. \tag{1.6}$$

This is just the Markov property in the sense of McKean [11]. We introduce the q-solutions mainly because we are not sure whether a solution of (1.4) is Markovian or not and our main interest concentrates on the Markovian solutions.

Instead of solving equation (1.4) directly, we associate it with the following martingale problem.

Definition 1.2. Let $u \in \mathscr{P}_1(E)$. $P \in \mathscr{P}(D, \mathscr{F})$ is called a solution of the martingale problem $[u, \Omega_{\|u_t\|}]$ if

(1) $\quad P \circ X_0^{-1} = u;$ (1.7)

(2) $\quad P \circ X_t^{-1} = u_t;$ (1.8)

(3) $\quad \forall j \in E, \ \left(I_{\{j\}}(X_t) - \int_0^t \Omega_{\|u_s\|} I_{\{j\}}(X_s)\,\mathrm{d}s, \mathscr{F}_t, P \right)$ is a martingale. (1.9)

The main results are as follows:

Theorem 1.1. *Assume* (1.1) *and* (1.2) *are satisfied. Then the solution of the martingale problem* $[u, \Omega_{\|u_t\|}]$ *for* $u \in \mathscr{P}_p(E)$ *exists.*

Theorem 1.2. *Assume* (1.1), (1.2) *and* (1.3) *are satisfied. Then* $\forall u \in \mathscr{P}_p(E)$ *the martingale problem* $[u, \Omega_{\|u_t\|}]$ *is well-posed. Thus the q-solution of* (1.4) *exists and is unique.*

The proofs of the theorems are given in Sections 2 and 3. The proof of Theorem 1.1 is based on the martingale approach proposed by Holley, Stroock and Varadhan [9, 14] combining the techniques from Funaki [7, 8] and Zhengs [16]. Recently Dawson and Zheng [5] gave another proof of the theorem, showing that a limit of the exchangeable N-cell system is a solution.

The phenomenon of phase transition is important in physics. Roughly speaking, it means that the number of stationary distributions of a system will change if certain parameters change. In Section 4, we show the existence of a phase transition for the second Schlögl model. It is given by:

Theorem 1.3. *Consider the second Schlögl model,*

$$A+2X \underset{\lambda_2}{\overset{\lambda_1}{\rightleftarrows}} 3X, \qquad X \underset{\lambda_4}{\overset{\lambda_3}{\rightleftarrows}} B.$$

(1) *If* $\lambda_1 < \lambda_2$, $\lambda_1 \leq \lambda_3 - 1$, *then the stationary distribution is unique.*
(2) *If*

$$(\lambda_3+1)^2 \leq \tfrac{1}{2} + \frac{\lambda_1+1}{3\lambda_3+\lambda_2+3}, \quad \lambda_3 > 0, \tag{1.10}$$

then there exists $C(\lambda_1, \lambda_2, \lambda_3) > 0$ *such that for each* $0 \leq \lambda_4 < C(\lambda_1, \lambda_2, \lambda_3)$ *there are at least three stationary distributions.*

2. Existence

Theorem 1.1 is the result of a series of lemmas which we will prove in the following. We first introduce two metrics on the space $\mathcal{P}(E)$ as follows:

$$\tilde{d}(u, v) = \sum_{j=0}^{\infty} 2^{-j}\{|u\{j\} - v\{j\}|\},$$
$$d(u, v) = \inf_{F \in \mathcal{P}(u,v)} \left\{ \int_{E\times E} |x-y| F(\mathrm{d}x, \mathrm{d}y) \right\}, \qquad u, v \in \mathcal{P}(E),$$

where $\mathcal{P}(u, v)$ is the set of all probability measures on $E\times E$ with marginals u and v.

Lemma 2.1. *Let* $\mathcal{C} \subset \mathcal{P}(D, \mathcal{F})$. *Assume for every* $T>0$, $0<\eta<1$, *there is a compact subset* B *of* E *such that*

$$\inf_{P\in\mathcal{C}} P\{X_t \in B: 0 \leq t \leq T\} > 1-\eta, \tag{2.1}$$

and for every $T>0$, $0<\varepsilon<1$,

$$\lim_{\delta\downarrow 0} \sup_{P\in\mathcal{C}} P\{\delta_\omega^T(\varepsilon) \leq \delta\} = 0, \tag{2.2}$$

where

$$\delta_\omega^T(\varepsilon) \triangleq \min\{\tau_n(\omega) - \tau_{n-1}(\omega): 1 \leq n \leq N_T(\omega)\},$$
$$N_T(\omega) \triangleq \min\{n: \tau_{n+1}(\omega) > T\},$$
$$\tau_0(\omega) = 0, \quad \tau_\omega(\omega) = \inf\{t \geq \tau_{n-1}: \rho(X_t, X_{\tau_{n-1}}) \geq \tfrac{1}{2}\varepsilon\}.$$

ρ *is the discrete metric on* E. *Then* $\mathcal{C}$ *is a precompact subset of* $\mathcal{P}(D, \mathcal{F})$ *in the weak topology.*

Proof. For $T>0$ let

$$W_\omega(\delta, T)=\inf_{\{t_i\}_{i=1,\ldots,n};\ r=1,2,\ldots}\left\{\sup_{t_{i-1}\leqslant s,t<t_i}\rho(X_t, X_s)\right\},$$

where $\{t_i\}$ satisfies $0=t_0<t_1<\cdots<t_{r-1}<T\leqslant t_r$ and also $\min_{0\leqslant i\leqslant r}(t_i-t_{i-1})\geqslant\delta>0$. By using Theorem 14.3 of [1], Theorem 1.1.4 of [14] and the fact that

$$\{\delta_\omega^T(\varepsilon)>\delta\}\subset\{W_\omega(\delta, T)\leqslant\varepsilon\},$$

we get the conclusion. □

In the following we always assume that the conditions (1.1) and (1.2) are satisfied.

Lemma 2.2. *For every $n\geqslant 1$, $u\in\mathscr{P}_1(E)$, there exists a $P_u^n\in\mathscr{P}(D,\mathscr{F})$ such that*

$$P_u^n\circ X_0^{-1}=u, \tag{2.3}$$

$$\text{for every } j\in E,\quad \left(I_j(X_t)-\int_0^t \Omega_{\|u_s^n\|}I_j(X_s)\,\mathrm{d}s,\ \mathscr{F}_t,\ P_u^n\right)\ \text{is a martingale,} \tag{2.4}$$

where $u_s^n=\tilde{u}^n_{[ns]/n}$, $\tilde{u}_s^n=P_u^n\circ X_s^{-1}$.

Proof. By applying Theorem 3.2 in [16], for every $u\in\mathscr{P}_1(E)$ and $i\in E$ there is a $P_{i,u}\in\mathscr{P}(D,\mathscr{F})$ such that

$$P_{i,u}\circ X_0^{-1}=\delta_i, \tag{2.5}$$

$$I_j(X_t)-\int_0^t \Omega_{\|u\|}I_j(X_s)\,\mathrm{d}s\ \text{ is a martingale with respect to } (\mathscr{F}_t, P_{i,u}). \tag{2.6}$$

For $s\in[0,+\infty)$, let Φ_s be the standard shift operator, $\Phi_s: D\to D$, $\Phi_s(\omega)(\cdot)=\omega(s+\cdot)$, $P_{i,u}^s$ be the unique probability measure on $(D,\mathscr{F})$ satisfying:

$$P_{i,u}^s\circ X_t^{-1}=\delta_i,\quad 0\leqslant t\leqslant s, \tag{2.7}$$

$$P_{i,u}^s\circ\Phi_s^{-1}=P_{i,u}. \tag{2.8}$$

For every $n\geqslant 1$ define $P_u^{n,1}\in\mathscr{P}(D,\mathscr{F})$ as follows:

$$P_u^{n,1}(\cdot)=\int_E P_{i,u}(\cdot)u(\mathrm{d}i). \tag{2.9}$$

By induction we define $P_u^{n,k+1}$ to be the unique probability measure satisfying

$$P_u^{n,k+1}=P_u^{n,k}\quad\text{on } \mathscr{F}_{k/n}. \tag{2.10}$$

A regular conditional probability distribution of $P_u^{n,k+1}$ given $\mathscr{F}_{k/n}$ is $P^{k/n}_{X_{k/n},u_{n,k}}$ where $u_{u,k}=P_u^{n,k}\circ X_{k/n}^{-1}$. (2.11)

Since $P_u^{n,k}\{k/n \le t\} \to 0$, as $k \to \infty$, there exists a unique probability measure $P_u^n \in \mathscr{P}(D, \mathscr{F})$ satisfying

$$P_u^n = P_u^{n,k} \quad \text{on } \mathscr{F}_{k/n}, \quad k = 1, 2, \ldots . \tag{2.12}$$

Then P_u^n is just what we want. □

Lemma 2.3. *For every $T > 0$, $u \in \mathscr{P}_p(E)$, we have*

$$\sup_n \|\tilde{u}_\cdot^n\|_{p,T} < \infty, \tag{2.13}$$

where $\|\tilde{u}_\cdot^n\|_{p,T} = \sup_{0 \le t \le T} \|\tilde{u}_t^n\|_p$, $\|\tilde{u}_t\|_p = \{\int x^p \tilde{u}_t^n(\mathrm{d}x)\}^{1/p}$.

Proof. For each integer $N > 0$ define $f_N(x) = x \wedge N$, $x \in E$ then $f_N(x) = \sum_{k=0}^N k I_{\{k\}}(x) + N(1 - \sum_{l=0}^N I_{\{l\}}(x))$.

Thus $(f_N(X_t) - \int_0^t \Omega_{\|u_s^n\|} f_N(X_s)\, \mathrm{d}s, \mathscr{F}_t, P_u^n)$ is a martingale by Lemma 2.2. This implies

$$\begin{aligned}
E^{P_u^n} f_N(X_t) &= E^{P_u^n} f_N(X_0) + E^{P_u^n} \int_0^t \Omega_{\|u_s^n\|} f_N(X_s)\, \mathrm{d}s \\
&\le \|u\| + E^{P_u^n}\left[\int_0^t \left(\sum_{k \ne 0} q_{X_s, X_s+k}(f_N(X_s + k) - f_N(X_s))\right.\right. \\
&\qquad \left.\left. + \|u_s^n\|(f_N(X_s + 1) - f_N(X_s))\right) \mathrm{d}s\right] \\
&\le \|u\| + E^{P_u^n} \int_0^t (M + \lambda X_s + \|u_s^n\|)\, \mathrm{d}s \\
&= \left(\|u\| + Mt + \int_0^t \|u_s^n\|\, \mathrm{d}s\right) + \lambda \int_0^t E^{P_u^n} X_s\, \mathrm{d}s.
\end{aligned} \tag{2.14}$$

Let $N \to \infty$, we have

$$E^{P_u^n} X_t \le \left(\|u\| + Mt + \int_0^t \|u_s^n\|\, \mathrm{d}s\right) + \lambda \int_0^t E^{P_u^n} X_s\, \mathrm{d}s. \tag{2.15}$$

By Gronwall's lemma, we have

$$E^{P_u^n} X_t \le \left(\|u\| + Mt + \int_0^t \|u_s^n\|\, \mathrm{d}s\right) \mathrm{e}^{\lambda t}. \tag{2.16}$$

This implies

$$E^{P_u^n} X_{(k+1)/n} \le \left(\|u_{k/n}^n\| + \frac{M}{n} + \int_{k/n}^{(k+1)/n} \|u_s^n\|\, \mathrm{d}s\right) \mathrm{e}^{\lambda/n}, \tag{2.17}$$

i.e.,

$$\|u_{(k+1)/n}^n\| \le \left(1 + \frac{1}{n}\right) \|u_{k/n}^n\|\, \mathrm{e}^{\lambda/n} + \frac{M}{n} \mathrm{e}^{\lambda/n}.$$

Since $\lambda > 1$, we get

$$\|u^n_{(k+1)/n}\| \leq \|u^n_{k/n}\| \, e^{2\lambda/n} + \frac{M}{n} e^{\lambda/n}. \tag{2.18}$$

By induction,

$$\|u^n_{(k+1)/n}\| \leq \|u\| \, e^{2(k+1)\lambda/n} + \frac{M}{n} \frac{e^{\lambda/n}}{e^{\lambda/n}+1} \cdot \frac{e^{2(k+2)\lambda/n}-1}{e^{\lambda/n}-1}. \tag{2.19}$$

Since $(k+1)/n \in [0, T]$, $n(e^{\lambda/n}-1) \to \lambda$ as $n \to \infty$ we get

$$\sup_n \|u^n_\cdot\|_{1,T} < \infty. \tag{2.20}$$

Now let $C(T) = \sup_n \|u^n_\cdot\|_{1,T}$, $g_N(x) = x^p \wedge N$. We also have $(g_N(X_t) - \int_0^t \Omega_{\|u^n_s\|} g_N(X_s)\, ds, \mathscr{F}_t, P^n_u)$ is a martingale. By (2.20) and condition (1.2) we have that

$$\begin{aligned}
\Omega_{\|u^n_s\|} g_N(X_s) &= \sum_{k \neq 0} q_{X_s, X_{s+k}} (g_N(X_s + k) - g_N(X_s)) \\
&\quad + \|u^n_s\| (g_N(X_{s+1}) - g_N(X_s)) \\
&\leq M + \lambda X^p_s + C(T)((X_s+1)^p - X^p_s) \\
&\leq M + \lambda X^p_s + C(T)(2^p - 1) X^p_s + 2^p \cdot C(T) \\
&= (M + 2^p \cdot C(T)) + (\lambda + C(T)(2^p - 1) X^p_s.
\end{aligned}$$

Similar to (2.14)-(2.18), we have that

$$\|\tilde{u}^n_t\|^p_p \leq \|u\|^p_p + (M + 2^p \cdot C(T)) t + \int_0^t (\lambda + C(T)(2^p - 1)) \|\tilde{u}^n_s\|^p_p \, ds.$$

By Gronwall's lemma, we get

$$\|\tilde{u}^n_t\|^p_p \leq (\|u\|^p_p + (M + 2^p C(T)) T) \, e^{(\lambda + C(T)(2^p - 1))T} \quad \forall t \in [0, T], \ n \geq 1.$$

Thus

$$\sup_n \|\tilde{u}^n_\cdot\|_{p,T} < \infty. \qquad \square$$

Lemma 2.4. *Let* $\sigma_N = \inf\{t \geq 0 \colon X_t \geq N\}$. *For every* $0 \leq T < +\infty$ *there exists a constant* $B(T)$ *such that*

$$\sup_n P^{(n)}_u \{\sigma_N < T\} \leq \frac{B(T)}{N}. \tag{2.21}$$

Proof. Let $f(x)=x$. B Lemma 2.3, for $0\leqslant T<+\infty$, there exists $M(T)>0$, $\lambda\geqslant 1$ such that

$$\Omega_{\|u_t^n\|}f(X)\leqslant M(T)+\lambda X \quad \forall n=1,2,\dots,\ t\in[0,T]. \tag{2.22}$$

Let

$$\phi(X)=X+M(T)/\lambda, \qquad \phi_m(x)=x\wedge m+M(T)/\lambda.$$

Then $(\phi_m(X_t)-\int_0^t \Omega_{\|u_s^n\|}\phi_m(X_s)\,\mathrm{d}s,\ \mathscr{F}_t,\ P_u^n)$ is a martingale, i.e., $\forall 0\leqslant t_1<t_2$, $A\in\mathscr{F}_{t_1}$, we have

$$\begin{aligned} E^{P_u^n}[\phi_m(X_{t_2});A] &= E^{P_u^n}[\phi_m(X_{t_1});A]+E^{P_u^n}\left[\int_{t_1}^{t_2}\Omega_{\|u_s^n\|}\phi_m(X_s)\,\mathrm{d}s;A\right]\\ &\leqslant E^{P_u^n}[\phi(X_{t_1});A]+E^{P_u^n}\left[\int_{t_1}^{t_2}(M(T)+\lambda X_s)\,\mathrm{d}s;A\right]\\ &= E^{P_u^n}[\phi(X_{t_1});A]+\lambda E^{P_u^n}\left[\int_{t_1}^{t_2}\phi(X_s)\,\mathrm{d}s;A\right]. \end{aligned}$$

This implies

$$E^{P_u^n}[\phi(X_{t_2});A]\leqslant E^{P_u^n}[\phi(X_{t_1});A]+\lambda E^{P_u^n}\left[\int_{t_1}^{t_2}\phi(X_s)\,\mathrm{d}s;A\right]. \tag{2.23}$$

By Gronwall's lemma we have

$$E^{P_u^n}[\phi(X_{t_2});A]\leqslant E^{P_u^n}[\phi(X_{t_1});A]\cdot \mathrm{e}^{\lambda(t_2-t_1)}. \tag{2.24}$$

This means that $Y_t=\phi(X_t)\,\mathrm{e}^{-\lambda t}$ is a supermartingale with respect to $(\mathscr{F}_t, P_u^n)$. Hence

$$\begin{aligned} \mathrm{e}^{-\lambda T}\cdot NP_u^n\{\sigma_N<T\} &\leqslant E^{P_u^n}\,\mathrm{e}^{-\lambda\cdot(T\wedge\sigma_N)}\phi(T\wedge\sigma_N)=E^{P_u^n}Y_{T\wedge\sigma_N}\\ &\leqslant E^{P_u^n}Y_0=\|u\|+M(T)/\lambda. \end{aligned}$$

Let $B(T)=(\|u\|+M(T)/\lambda)\,\mathrm{e}^{\lambda T}$, we get (2.21). □

Lemma 2.5. *For each $u\in\mathscr{P}_1(E)$, $\{P_u^n\}_{n\geqslant 1,2,\dots}$ is a precompact subset of $\mathscr{P}(D,\mathscr{F})$ in the weak topology.*

Proof. For any $L>0$ let $B_L=\{0,1,2,\dots,L\}$, $\sigma_L=\inf\{t: X_t\geqslant L\}$. Then we have, by Lemma 2.4, that

$$\inf_n P_u^n\{X_t\in B_L: 0\leqslant t\leqslant T\}\geqslant \inf_n P_u^n\{\sigma_L\geqslant T\}\to 1, \quad L\to\infty. \tag{2.25}$$

Let $A_L=\{\sup_{0\leqslant t\leqslant T}X_t\leqslant L\}$. Then

$$P_u^n\{\delta_\omega^T(\varepsilon)<\delta\}\leqslant P_u^n\{\delta_\omega^T(\varepsilon)<\delta, A_L\}+P_u^n(A_L^{\mathrm{c}}). \tag{2.26}$$

By (2.25) $\sup_n P_u^n\{A_L^c\} = \gamma_L \to 0$ as $L \to \infty$. On the other hand

$$P_u^n\{\delta_\omega^T(\varepsilon) < \delta, A_L\} \leqslant P_u^n\{\delta_\omega^T(\varepsilon) < \delta, A_L, N_T(\omega) \leqslant k\} + P^n\{A_L, N_T(\omega) > k\}, \tag{2.27}$$

$$\begin{aligned}
&P_u^n\{\delta_\omega^T(\varepsilon) < \delta, A_L, N_T(\omega) \leqslant k\} \\
&= \sum_{i=1}^{k} P_u^n\{\delta_\omega^T(\varepsilon) < \delta, A_L, N_T(\omega) = i\} \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{i} P_u^n\{\tau_j - \tau_{j-1} < \delta, A_L, N_T(\omega) = i\} \\
&\leqslant \sum_{i=1}^{i} \sum_{j=1}^{i} P_u^n\left\{\tau_j - \tau_{j-1} < \delta, \bigvee_{l=1}^{j-1} X_{\tau_l} \leqslant L, \tau_{j-1} \leqslant T\right\} \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{i} E^{P_u^n}\left\{E^{P_{j-1}^n}\left[I_{\{X_{\tau_{j-1}}\}^c}(X_{\tau_j \wedge (\tau_{j-1}+\delta)}) = 1, \bigvee_{l=1}^{j-1} X_{\tau_l} \leqslant L, \tau_{j-1} \leqslant T\right]\right\} \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{i} E^{P_u^n}\left\{E^{P_{j-1}^n}\left[\int_{\tau_{j-1}}^{\tau_j \wedge (\tau_{j-1}+\delta)} \Omega_{\|u_s^n\|} I_{\{X_{\tau_{j-1}}\}^c}(X_s)\, \mathrm{d}s\right];\right. \\
&\qquad\qquad \left.\bigvee_{l=1}^{j-1} X_{\tau_l} \leqslant L, \tau_{j-1} \leqslant T\right\} \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{i} E^{P_u^n}\left\{E^{P_{j-1}^n}\left[-\int_{\tau_{j-1}}^{\tau_j \wedge (\tau_{j-1}+\delta)} \Omega_{\|u_s^n\|} I_{\{X_{\tau_{j-1}}\}}(X_s)\, \mathrm{d}s\right];\right. \\
&\qquad\qquad \left.\bigvee_{l=1}^{j-1} \leqslant L, \tau_{j-1} \leqslant T\right\} \\
&\leqslant \sum_{i=1}^{k} \sum_{j=1}^{i} C(L, T)\delta = \tfrac{1}{2}k(k+1)C(L, T)\delta,
\end{aligned} \tag{2.28}$$

where P_{j-1}^n is the r.c.p.d. of P_u^n given $\mathscr{F}_{\tau_{j-1}}$, $C(L, T) = \sup_{0 \leqslant j \leqslant L}(|q_{j,j}| + \sup_{k \in E, k \neq j} q_{k,j}) + \sup_n \|u_\cdot^n\|_{1,T} < \infty$ by (1.1).

$$\begin{aligned}
&P_u^n\{A_L, N_T(\omega) > k\} \\
&= P_u^n\{\tau_k \leqslant T, A_L\} \\
&\leqslant E^{P_u^n}[\mathrm{e}^{T-\tau_k}; \tau_k \leqslant T, A_L] \\
&\leqslant \mathrm{e}^T E^{P_u^n}\{E^{P_{k-1}^n}[\mathrm{e}^{-(\tau_k - \tau_{k-1})}; \tau_k \leqslant T, A_L]\,\mathrm{e}^{-\tau_{k-1}}\} \\
&\leqslant \mathrm{e}^T E^{P_u^n}\{\mathrm{e}^{-\tau_{k-1}}[\mathrm{e}^{-t_0} P_{k-1}^n\{\tau_k - \tau_{k-1} > t_0, \tau_k \leqslant T, A_L\} \\
&\qquad + P_{k-1}^n(\tau_k - \tau_{k-1} \leqslant t_0, \tau_k \leqslant T, A_L)]\}
\end{aligned}$$

$$\leqslant e^T E^{P_u^n}\Big\{[e^{-\tau_{k-1}} e^{-t_0} + (1-e^{-t_0}) P_{k-1}^n(\tau_k - \tau_{k-1} \leqslant t_0)];$$
$$\tau_{k-1} \leqslant T, \bigvee_{l=1}^{k-1} X_{\tau_l} \leqslant L\Big\}$$
$$\leqslant e^T E^{P_u^n}\Big\{e^{-\tau_{k-1}}(e^{-t_0} + (1-e^{-t_0}) C(L, T) t_0);$$
$$\tau_{k-1} \leqslant T, \bigvee_{l=1}^{k-1} X_{\tau_l} \leqslant L\Big\}. \tag{2.29}$$

Let $\psi(t) = e^{-t} + (1-e^{-t}) C(L, T) t$.

$$\psi(0) = 1,$$
$$\psi'(t)|_{t=0} = [-e^{-t} + (1-e^{-t}) C(L, T) + t\, e^{-t} \cdot C(L, T)]|_{t=0} = -1 < 0.$$

Thus for fixed L and T, there is a $t_0 \in (0, 1)$ such that $0 < \psi(t_0) < 1$.

Denote this $\psi(t_0)$ by $\Gamma(L, T)$. By induction, we have

$$P_u^n\{A_L, N_T(\omega) > k\} \leqslant \Gamma(L, T)^k\, e^T. \tag{2.30}$$

Combining (2.25)-(2.30), we get

$$\sup_n P_u^n\{\delta_\omega^T(\varepsilon) < \delta\} \leqslant \tfrac{1}{2} k(k+1) C(L, T)\delta + \Gamma(L, T)^k\, e^T + \gamma_L. \tag{2.31}$$

In (2.31) let $\delta \to 0$, then $k \to \infty$. Finally let $L \to \infty$. Then the following is true.

$$\lim_{\delta\to 0} \sup_n P_u^n\{\delta_\omega^T(\varepsilon) < \delta\} = 0. \tag{2.32}$$

By applying Lemma 2.1 we get the result. □

Lemma 2.6. (a) $\forall 0 \leqslant T < \infty$, $\lim_{N\to\infty} \sup_{n, t\in[0,T]} \int_{\{x>N\}} x \tilde{u}_t^n(dx) = 0$

(b) $\forall \varepsilon > 0$, $T \in [0, +\infty)$, $\exists \delta(\varepsilon, T) > 0 \ni \forall s, t \in [0, T], |s-t| < \delta(\varepsilon, T)$, *we have* $\sup_n \tilde{d}(\tilde{u}_t^n, \tilde{u}_s^n) < \varepsilon$.

Proof. (a) This is a direct result of Lemma 2.3 and Hölder's inequality.

(b) $$\tilde{d}(\tilde{u}_t^n, \tilde{u}_s^n) = \sum_{j=0}^{\infty} 2^{-j}[|E^{P_u^n}(I_j(X_t) - I_j(X_s))|].$$

$$E^{P_u^n}(I_j(X_t) - I_j(X_s)) = E^{P_u^n}\left[\int_s^t \Omega_{\|u_\tau^n\|} I_j(X_\tau)\, d\tau\right]$$
$$= E^{P_u^n} \int_s^t \Bigg[\sum_{k\neq 0} q_{X_\tau, X_\tau+k}(I_j(X_\tau + k) - I_j(X_\tau))$$
$$+ \|u_\tau^n\|(I_j(X_\tau + 1) - I_j(X_\tau))\Bigg]\, d\tau$$
$$= \int_s^t \Bigg\{\sum_{k\neq 0} [q_{j-k,j} P_u^n(X_\tau = j-k) - q_{j,j+k} P_u^n(X_\tau = j)]$$
$$+ \|u_\tau^n\|(P_u^n(X_\tau = j-1) - P_u^n(X_\tau = j))\Bigg\}\, d\tau, \tag{2.33}$$

$\forall \varepsilon > 0,\ T > 0,\ \exists N > 0 \ni \sum_{j=N+1}^{\infty} 2^{-j} < \frac{1}{2}\varepsilon$. For $j = 0, 1, \ldots, N$, (2.33) becomes

$$|E^{P_u^n}(I_j(X_t) - I_j(X_s)| \leq C(N, T)(t-s). \tag{2.34}$$

Let $\delta = \varepsilon/(2C(N, T))$, for $|t-s| < \delta$, $t, s \in [0, T]$ we get that

$$\tilde{d}(\tilde{u}_t^n, \tilde{u}_s^n) \leq \tfrac{1}{2}\varepsilon + \sum_{j=1}^{N} 2^{-j}(|C(N, T)(t-s)|) < \varepsilon. \quad \square$$

This lemma shows that $\{\tilde{u}_\cdot^n\}_{n=1}^{\infty}$ is a family of $\mathscr{P}(E)$-valued $\tilde{d}$-equicontinuous functions and the set $\{\tilde{u}_t^n\colon n = 1, 2, \ldots, t \in [0, T]\}$ is precompact in both the space $(\mathscr{P}(E), d)$ and the space $(\mathscr{P}(E), \tilde{d})$.

Lemma 2.5 and Lemma 2.6 together implies the following:

Lemma 2.7. *There exists a subsequence $\{u_\cdot^{n_k}\}$ of $\{u_\cdot^n\}$ and $\{P_u^{n_k}\}$ of $\{P_u^n\}$ such that $P_u^{n_k}$ converges weakly to a $P_u \in \mathscr{P}(D, \mathscr{F})$ and $d(u_t^{n_k}, u_t) \to 0$ $(k \to \infty)$ where $u_t = P_u \circ X_t^{-1}$.*

Proof. The proof is similar to that used in the proof of Step 5 of Theorem 2.1 in [7]. The key part is

$$\tilde{d}(u_t^{n_k}, u_t) \leq \sup_{0 \leq s \leq t} \tilde{d}(\tilde{u}_s^{n_k}, u_s) + \tilde{d}(u_{[n_k t]/n_k}, u_t) \to 0, \quad \text{as } k \to \infty. \quad \square$$

Proof of Theorem 1.1. Let

$$\Psi_t = \int_0^t \Omega_{\|u_s\|} I_j(X_s)\, \mathrm{d}s, \qquad \Psi_t^n = \int_0^t \Omega_{\|u_s^n\|} I_j(X_s)\, \mathrm{d}s.$$

Define $T_J = \{t \geq 0\colon P_u\{X_t \neq X_{t-}\} = 0\}$. Then T_J is dense in $[0, \infty)$ and T_J^c is a Borel set with zero measure.

By Lemma 2.7, for each $t \in T_j$ and $\Phi \in C_b(D, \mathscr{F})$ (here $C_b(D, \mathscr{F})$ is the set of all bounded continuous functions on D), there exist $\{n_k\} \subset \{n\}$ such that

$$E^{P_u^{n_k}}[\Phi \cdot I_j(X_t)] \to E^{P_u}[\Phi \cdot I_j(X_t)], \tag{2.35}$$

$$|E^{P_u^{n_k}}[(\Psi_t^{n_k} - \Psi_t)\Phi]| \leq \sup \Phi \int_0^t d(u_s^{n_k}, u_s)\, \mathrm{d}s \to 0 \quad (n_k \to \infty). \tag{2.36}$$

Since Φ is bounded and condition (1.1) is assumed, $|(\Omega_{\|u_s\|} I_j(X_s)) \cdot \Phi|$ is a bounded continuous function of $\omega \in D$ except a zero measure set T_J^c. This implies

$$E^{P_u^{n_k}}[\Psi_t \cdot \Phi] \to E_u^P[\Psi_t \cdot \Phi] \quad \text{for } t \in [0, \infty). \tag{2.37}$$

(2.35)–(2.37) and Lemma 2.2 imply that

$$\forall 0 \leq t_1 < t_2, \quad E^{P_u}[(I_j(X_{t_2}) - \Psi_{t_2})\Phi] = E^{P_u}[(I_j(X_{t_1}) - \Psi_{t_1})\Phi], \tag{2.38}$$

for $t_1, t_2 \in T_J$, $\Phi \in C_b(D, \mathscr{F})$.

By the right-continuity of $I_j(X_t) - \Psi_t$ and the dominated convergence theorem, we get that P_u is a solution of the martingale problem $[u, \Omega_{\|u_t\|}]$ for $u \in \mathscr{P}_p(E)$. $\square$

Corollary. *Assume* (1.1) *and* (1.2) *are satisfied.* P_u *is a solution of the martingale problem* $[u, \Omega_{\|u_s\|}]$ *with* $u \in \mathcal{P}_p(E)$. *Then* $u_t = P_u \circ X_t^{-1}$ *is a continuous function from* $[0, \infty)$ *to* $(\mathcal{P}(E), d)$.

Proof. Similar to the proof of Lemma 2.3 and Lemma 2.6 we get the following:

$$\sup_{t \in [0,T]} \|u_t\|_p < \infty, \tag{2.39}$$

$$\lim_{N \to \infty} \sup_{0 \leq t \leq T} \int_{\{x > N\}} x u_t(\mathrm{d}x) = 0, \tag{2.40}$$

$$\sup_{\substack{t,s \in [0,T] \\ |t-s| < \delta}} \tilde{d}(u_t, u_s) \to 0 \quad \text{as } \delta \to 0. \tag{2.41}$$

(2.40) shows that $\{u_t: 0 \leq t \leq T\}$ is precompact in the space $(\mathcal{P}(E), d)$. (2.41) shows that u_t is uniformly continuous in $[0, T]$ in the space $(\mathcal{P}(E), \tilde{d})$. They together imply the continuity of u_t in $(\mathcal{P}(E), d)$. □

3. Uniqueness

This section is devoted to the proof of Theorem 1.2. We will finish the proof by three steps:

(a) Any q-solution of (1.4) is a solution of the martingale problem $[u, \Omega_{\|u_t\|}]$ for $u \in \mathcal{P}_p(E)$.

(b) The martingale problem $[u, \Omega_{\|u_t\|}]$ is well-posed.

(c) The unique solution in (b) is the q-solution.

Definition 3.1. Let $u \in \mathcal{P}_1(E)$, $u_t \in \mathcal{P}_1(E)$, $t > 0$. $P_{u,u.} \in \mathcal{P}(D, \mathcal{F})$ is called a solution of $\langle u., u \rangle$ if

$$P_{u,u.} \circ x_0^{-1} = u, \tag{3.1}$$

$$\forall j \in E, \quad \left(I_j(X_t) - \int_0^t \Omega_{\|u_s\|} I_j(X_s)\, \mathrm{d}s, \mathcal{F}_t, P_{u,u.} \right) \text{ is a martingale.} \tag{3.2}$$

Lemma 3.1. *Let* $u \in \mathcal{P}_p(E)$. $P_u \in P(D, \mathcal{F})$ *is a q-solution of* (1.4) *with initial distribution* u *implies that* P_u *is also a solution of the martingale problem* $[u, \Omega_{\|u_t\|}]$.

Proof. Let P_u be a q-solution of (1.4) with initial distribution u. For any fixed $j \in E$, let $Y_t^{(j)} = I_j(X_t) - \int_0^t \Omega_{\|u_s\|} I_j(X_s)\, \mathrm{d}s$. Then $\forall t \geq 0$, $s \geq 0$,

$$\begin{aligned} &E^{P_u}[Y_{t+s}^{(j)} - Y_t^{(j)} \mid \mathcal{F}_t] \\ &= \int_0^s \left(\frac{\mathrm{d}p(t, X_t;\, t+\tau, j)}{\mathrm{d}\tau} - \sum_{k \in E} p(t, X_t;\, t+\tau, k) \Omega_{\|u_{t+\tau}\|} I_j(k) \right) \mathrm{d}\tau = 0. \end{aligned}$$

Thus P_u is a solution for the martingale problem $[u, \Omega_{\|u_t\|}]$. □

Lemma 3.2. *Assume* (1.1), (1.2) *and* (1.3) *are satisfied. Then for each* $u \in \mathscr{P}_p(E)$, *the solution* $\langle u., u\rangle$ *is unique.*

Proof. Theorem 3.2 in [16] proves that the solution $P_{\delta_i,u.}$ of $\langle u., \delta_i\rangle$ for each $i \in E$ is unique and also $P_{\delta_i,u.}\{X_t = j\} \triangleq p(0, i; t, j)$ is the minimal nonnegative solution [10] of the equation

$$\phi(0, i; t, j) = \delta_{ij}\, \mathrm{e}^{-\int_0^t q(\tau,j)\,\mathrm{d}t} + \int_0^t \sum_{k\neq j} \phi(0, i; t, k) q_{k,j}(\tau)\, \mathrm{e}^{-\int_\tau^t q(\sigma,j)\,\mathrm{d}\sigma}\, \mathrm{d}\tau. \tag{3.3}$$

Here $(q_{k,j}(t))_{k,j\in E}$ corresponds to $\Omega_{\|u_t\|}$, $q(t, j) = -q_{j,j}(t)$.

Let $P_{u,u.}(\cdot) = \sum_{i\in E} u(\{i\})P_{\delta_i,u.}(\cdot)$. Then $P_{u,u.}$ is a solution of $\langle u., u\rangle$.

By the comparison theorem (see [10]), $\phi_u(t, j) = P_{u.,u}(X_t = j)$ is the minimal nonegative solution of the equation

$$\phi_u(t, j) = u(\{j\})\, \mathrm{e}^{-\int_0^t q(\tau,j)\,\mathrm{d}\tau} + \int_0^t \sum_{k\neq j} \phi_u(\tau, k) q_{k,j}(\tau)\, \mathrm{e}^{-\int_\tau^t q(\sigma,j)\,\mathrm{d}\sigma}\, \mathrm{d}\tau. \tag{3.4}$$

Let P_u be a solution of $\langle u., u\rangle$, $P_u(t, j) \triangleq P_u(X_t = j)$. Then we have

$$P_u(t, j) = u(\{j\}) + \int_0^t \sum_{k=0}^{\infty} P_u(s, k) q_{k,j}(s)\, \mathrm{d}s, \tag{3.5}$$

which implies

$$P_u(t, j) = u(\{j\})\, \mathrm{e}^{-\int_0^t q(\tau,j)\,\mathrm{d}\tau} + \int_0^t \sum_{k\neq j} P_u(\tau, k) q_{k,j}(\tau)\, \mathrm{e}^{-\int_0^t q(\sigma,j)\,\mathrm{d}\sigma}\, \mathrm{d}\tau. \tag{3.6}$$

This means $\phi_u(t, j) \leq P_u(t, j)$. Since $\sum_{j\in E} \phi_u(t, j) = 1$, we have $\phi_u(t, j) \equiv P_u(t, j)$.

By the uniqueness of the solution of $\langle u., \delta_i\rangle$ of $i \in E$ and Theorem 1.2.10 [14], we can use the same argument to get

$$P_u(X_t = j \mid X_s) = p(s, X_s; t, j), \quad P_u\text{-a.s.} \tag{3.7}$$

Thus the solution of $\langle u., u\rangle$ is unique. □

Lemma 3.3. *Assume* (1.1), (1.2) *and* (1.3) *are satisfied. Then the martingale problem* $[u, \Omega_{\|u_t\|}]$ *is well-posed for* $u \in \mathscr{P}_p(E)$.

Proof. By Theorem 1.1 it needs only to prove the uniqueness of the problem.

Let $P_u^1, P_u^2 \in \mathscr{P}(D, \mathscr{F})$ be any two solutions of $[u, \Omega_{\|u_t\|}]$. $u_t^l = P^l \circ x_t^{-1}$, $u_0^l = u$, $l = 1, 2$.

Define

$$Af(j_1, j_2) = \sum_{k\neq 0} (q_{j_1,j_1+k} - q_{j_2,j_2+k})^+ \cdot (f(j_1+k, j_2) - f(j_1, j_2))$$
$$+ \sum_{k\neq 0} (q_{j_2,j_2+k} - q_{j_1,j_1+k})^+ \cdot (f(j_1, j_2+k) - f(j_1, j_2))$$
$$+ \sum_{k\neq 0} q_{j_1,j_1+k} \wedge q_{j_2,j_2+k} \cdot (f(j_1+k, j_2+k) - f(j_1, j_2)), \tag{3.8}$$

$$D_t f(j_1, j_2) = (\|u_t^1\| - \|u_t^2\|)^+ \cdot (f(j_1+1, j_2) - f(j_1, j_2))$$
$$+ (\|u_t^2\| = \|u_t^1\|)^+ \cdot (f(j_1, j_2+1) - f(j_1, j_2))$$
$$+ \|u_t^1\| \wedge \|u_t^2\| \cdot (f(j_1+1, j_2+1) - f(j_1, j_2)), \tag{3.9}$$

$$A_t f(j_1, j_2) = Af(j_1, j_2) + D_t f(j_1, j_2)$$
$$= \sum_{(j_1,j_2)\in E\times E} q_{(i_1,i_2),(j_1,j_2)}(t)(f(j_1, j_2) - f(i_1, i_2)). \tag{3.10}$$

Let $(P(t, (i_1, i_2), (j_1, j_2)))_{(i_1,i_2),(j_1,j_2)\in E\times E}$ be the minimal A_t-processes [2].

Define

$$F(t) = \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(j_1,j_2)\in E\times E} P(t, (i_1, i_2), (j_1, j_2))|j_2 - j_1|. \tag{3.11}$$

Where F is a probability measure on $E\times E$ with marginals u_0^1 and u_0^2. Then by the Kolmogorov's forward differential equation and (1.1), (1.2) and (1.3), we have

$$\frac{dF(t)}{dt} = \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(j_1,j_2)\in E\times E} \frac{d}{dt} P(t, (i_1, i_2), (j_1, j_2))|j_2 - j_1|$$
$$= \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(j_1,j_2)\in E\times E} \sum_{(l_1,l_2)\in E\times E} P(t, (i_1, i_2), (l_1, l_2)) \cdot q_{(l_1,l_2),(j_1,j_2)}(t)|j_2 - j_1|$$
$$= \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(l_1,l_2)\in E\times E} \sum_{(j_1,j_2)\in E\times E} P(t, (i_1, i_2), (l_1, l_2)) \cdot q_{(l_1,l_2),(j_1,j_2)}(t)|j_2 - j_1|$$
$$= \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(l_1,l_2)\in E\times E} P(t, (i_1, i_2), (l_1, l_2)) \cdot \sum_{(j_1,j_2)\in E\times E} q_{(l_1,l_2),(j_1,j_2)}(t)|j_2 - j_1|$$
$$\leqslant \sum_{(i_1,i_2)\in E\times E} F\{(i_1, i_2)\} \sum_{(l_1,l_2)\in E\times E} P(t, (i_1, i_2), (l_1, l_2)) \cdot (C|l_1 - l_2| + d(u_t^1, u_t^2))$$
$$= C\cdot F(t) + d(u_t^1, u_t^2), \tag{3.12}$$

which implies

$$d(u_t^1, u_t^2) \leqslant e^{Ct}F(0) + \int_0^t e^{C(t-s)} d(u_s^1, u_s^2)\, ds. \tag{3.13}$$

Now let F be

$$F\{(i,j)\}=\begin{cases}0, & i\neq j,\\ u(\{i\}), & i=j.\end{cases}$$

Thus we have

$$d(u_t^1, u_t^2)\leq \int_0^t e^{C(t-s)} d(u_s^1, u_s^2)\,ds. \tag{3.14}$$

Therefore $u_t^1=u_t^2$ for all $t\geq 0$ and the conclusion follows from Lemma 3.2. □

Proof of Theorem 1.2. We need only to prove that the unique solution obtained in Lemma 3.3 is a q-solution. Any solution of the martingale problem $[u, \Omega_{\|u_t\|}]$ for $u\in\mathscr{P}_p(E)$ must satisfy equation (1.4). Since the unique solution obtained in Lemma 3.3 is just the minimal q-processes [10] with the infinitesimal generator $\Omega_{\|u_t\|}$. Combining with Lemma 3.1-3.3, we reach the conclusion. □

4. Stationary distributions of the birth and death models

In this section, we deal only with birth-death models and assume that (1.1), (1.2) and (1.3) are satisfied.

Let $a_1=q_{i,i-1}$, $b_i=q_{i,i+1}$ and assume that they are polynomials of some order. Define

$$Q=\begin{pmatrix}-b_0 & b_0 & 0 & 0 & \cdots\\ a_1 & -(a_1+b_1) & b_1 & 0 & \cdots\\ 0 & a_2 & -(a_2+b_2) & b_2 & 0\\ \vdots & & \ddots & \ddots & \vdots\end{pmatrix}, \tag{4.1}$$

$$Q_\lambda=Q+\begin{pmatrix}-\lambda & \lambda & 0 & 0 & \cdots\\ 0 & -\lambda & \lambda & 0 & \cdots\\ \vdots & & \ddots & \ddots & \end{pmatrix}. \tag{4.2}$$

Definition 4.1. $u\in\mathscr{P}_p(E)$ is called a stationary distribution of the q-solution of (1.4) if $P\circ X_0^{-1}=u$ implies that for all $t\geq 0$, $P\circ X_t^{-1}=u$.

Let U denote the set of all stationary distributions of the q-solution of (1.4). We have

Lemma 4.1. (1) *For each $u\in U$, we have*

$$(u_0, u_1, \ldots)Q_{\|u\|}=0, \quad \textit{where } u_i=u(\{i\}). \tag{4.3}$$

(2) *Assume $u\in\mathscr{P}_p(E)$ and (4.3) is satisfied, then $u\in U$.*

Proof. (1) Let P_u be a q-solution of (1.4) with initial distribution u. $(q_{i,j}(t))_{i,j\in E}=Q_{\|u_t\|}$. For $u\in U_p=\mathscr{P}_p(E)\cap U$, we have

$$0=\frac{P_u(X_t=j)-u(\{j\})}{t}=\frac{1}{t}\int_0^t\sum_{k\in E}P_u(X_s=k)q_{i,j}(s)\,\mathrm{d}s. \tag{4.4}$$

Let $t\to 0$, we get (4.3).

(2) Assume $u\in\mathscr{P}_p(E)$, $(u_0,u_1,u_2,\ldots)Q_{\|u\|}=0$. Let $(P(t,i,j))_{i,j}\in E$ be the minimal $Q_{\|u\|}$-process, then

$$P(t,i,j)=\delta_{ij}+\int_0^t\sum_{k\in E}q_{\|u\|}(i,k)P(s,k,j)\,\mathrm{d}s. \tag{4.5}$$

This implies

$$\sum_{i\in E}u_iP(t,i,j)=u_j+I, \tag{4.6}$$

$$\begin{aligned}I&=\sum_{i\in E}u_i\int_0^t\sum_{k\in E}q_{\|u\|}(i,k)P(s,k,j)\,\mathrm{d}s\\&=\int_0^t\sum_{k\in E}\sum_{i\in E}P(s,k,j)u_iq_{\|u\|}(i,k)\,\mathrm{d}s=0,\end{aligned} \tag{4.7}$$

where $(q_{\|u\|}(i,k))_{i,k\in E}=Q_{\|u\|}$.

From the construction of P_u^n and the above argument, we can get

$$P_u^n\circ X_t^{-1}=u\quad\text{for } t\in[0,1/n]. \tag{4.8}$$

By induction, we get that

$$P_u^n\circ X_t^{-1}=u\quad\text{for } t\geqslant 0. \tag{4.9}$$

By Lemma 2.7 and the corollary in Section 2, we have

$$P_u\circ X_t^{-1}=u,\ \forall t\geqslant 0,\quad\text{i.e. } u\in U.\qquad\square \tag{4.10}$$

Theorem 4.2. *Assume*

$$L_\lambda=1+\sum_{i=1}^{\infty}\frac{(b_0+\lambda)\cdots(b_{i-1}+\lambda)}{a_1\cdots a_i}<\infty\quad\textit{for every } \lambda\geqslant 0, \tag{4.11}$$

then (a) *If* $|U|\geqslant 1$, *then the equation*

$$f(\lambda)=\lambda+\sum_{i=1}^{\infty}(\lambda-i)\frac{(b_0+\lambda)\cdots(b_{i-1}+\lambda)}{a_1\cdots a_i}=0 \tag{4.12}$$

has at least one nonnegative solution, where $|U|$ *denotes the number of elements in U.*

(b) *Assume* $f(\lambda_0)=0$, $\lambda_0\geqslant 0$; *take* u_{λ_0} *to be*

$$u_{\lambda_0}=\frac{1}{L_{\lambda_0}}\left(1,\ldots\frac{(b_0+\lambda)\cdots(b_{i-1}+\lambda)}{a_1\cdots a_i},\ldots\right). \tag{4.13}$$

If $u_{\lambda_0}\in\mathscr{P}_p(E)$, *then* $u_{\lambda_0}\in U_p$.

Proof. Solve the equation (4.3), we get (a).

For the proof of (b), notice that u_{λ_0} is defined to satisfy the equation (4.5). Thus by Lemma 4.1(2), we get that $u_{\lambda_0} \in U_p$. □

Theorem 4.3. *Assume that the order of the polynomial a_i is larger than that of b_i which is larger than one, the $|U| < \infty$.*

Proof. We only prove the case when the order of a_i is equal to that of b_i plus one. (Other cases are similar.)

Let

$$\begin{aligned} a_i &= \alpha_0 + \cdots + \alpha_m i^m, \quad \alpha_m > 0, \\ b_i &= \beta_0 + \cdots + \beta_{m-1} i^{m-1}, \quad \beta_{m-1} > 0, \end{aligned} \qquad m \geq 2.$$

When i is large enough, there exists a constant $r > 0$ such that $a_i > r \cdot i^m$. For simplicity, we only prove the case for $a_i = r \cdot i^m$.

Under these restrictions, we have

$$f(\lambda) = \lambda + \sum_{i=1}^{[\lambda]} (\lambda - i) \frac{(b_0 + \lambda) \cdots (b_{i-1} + \lambda)}{a_1 \cdots a_i} - I, \tag{4.14}$$

where $[\lambda]$ is the integer part of λ.

$$\begin{aligned} I &= \sum_{i=[\lambda]+1}^{\infty} (i - \lambda) \frac{(b_0 + \lambda) \cdots (b_{i-1} + \lambda)}{a_1 \cdots a_i} \\ &\leq \sum_{i=[\lambda]+1}^{\infty} (i - \lambda) \frac{(b_0 + \lambda) \cdots (b_{i-1} + \lambda)}{r^i (i!)^m}. \end{aligned} \tag{4.15}$$

Let $\beta = \sum_{i=0}^{m-1} |\beta_i|$, $\tilde{\beta} = \max\{1, \beta\}$, $k = \max\{n\colon n \in N, n^{m-1} \leq \lambda\}$. Then

$$\begin{aligned} 0 \leq T &\leq \sum_{i=[\lambda]+1}^{\infty} (\tilde{\beta} + i) \frac{\tilde{\beta}^i (1 + (k+1)^{m-1}) \cdots (k^{m-1} + (k+1)^{m-1})((k+1) \cdots i)^{m-1}}{2^{k-i} r^i (i!)^m} \\ &\leq \sum_{i=[\lambda]+1}^{\infty} \left(\frac{2k+1}{k+1} \right)^{m-1} (\tilde{\beta} + i) \frac{(2k)^{2m-2}}{2k} \frac{(2\tilde{\beta}/r)^i}{i!} \\ &\leq \sum_{i=[\lambda]+1}^{\infty} 2^{m-1} (\tilde{\beta} + i) \frac{(4^{m-1} \tilde{\beta}/r)^i}{i!} \to 0, \quad \lambda \to \infty. \end{aligned} \tag{4.16}$$

Therefore, there exists $\sigma > 0$ such that for all $\lambda \geq \sigma$, $f(\lambda) > 0$, i.e., the nonnegative solutions of $f(\lambda) = 0$ belongs to $[0, \sigma)$.

By Rolle's theorem and the absolute convergence of $f(\lambda)$, it is clear that the number of nonnegative solutions of $f(\lambda) = 0$ is finite. But Theorem 4.2 shows that the number of nonnegative solutions of $f(\lambda) \geq 0$ is not less than $|U|$. Hence $|U| < \infty$. □

Theorem 4.4. *Assume* $C<-1$. *Then* $|U|\leq 1$.

Proof. For any $u_1, u_2 \in U$, by applying (3.13), we have

$$d(u^1,u^2)\leq e^{Ct}d(u^1,u^2)+\int_0^t e^{C(t-s)}d(u^1,u^2)\,ds, \tag{4.17}$$

which implies

$$\left(1+\frac{1}{C}\right)d(u^1,u^2)\leq\left(1+\frac{1}{C}\right)d(u^1,u^2)\cdot e^{Ct}. \tag{4.18}$$

By assumption, we get

$$d(u^1,u^2)=0, \quad \text{i.e. } |U|\leq 1. \qquad \square \tag{4.19}$$

Proof of Theorem 1.3. For Schlögl model, p can be chosen as 2 and every u_λ defined by (4.13) belongs to $P_p(E)$, so $|U_2|$ equals the number of the roots of $f(x)=0$.

(1) If $\lambda_1<\lambda_2$, $\lambda_1\leq\lambda_3-1$, then

$$C=\sup_{j_2>j_1\geq 0}\{\tfrac{1}{2}(\lambda_1+\lambda_2)(j_1+j_2)-\tfrac{1}{2}\lambda_1-\tfrac{1}{3}\lambda_2-\tfrac{1}{6}\lambda_2(j_1^2+j_1\cdot j_2+j_2^2)\}<-1.$$

Thus the result follows from Theorem 4.4.

(2) We consider the following two cases:

(a) $\lambda_4=0$. In this case

$$a_i=\lambda_2\binom{i}{3}+(\lambda_3+1)i, \qquad b_i=\lambda_1\binom{i}{2},$$

$$\begin{aligned} f(\lambda)&=\lambda\cdot g(\lambda), \quad g(\lambda)\\ &=1+\frac{\lambda-1}{1+\lambda_3}+\frac{\lambda(\lambda-2)}{2(1+\lambda_3)^2}+\frac{\lambda(\lambda-3)(\lambda+\lambda_1)}{2(1+\lambda_3)^2(3\lambda_3+\lambda_2+3)}\\ &\quad+\sum_{i=4}^{\infty}(\lambda-i)\frac{(b_0+\lambda)\cdots(b_{i-1}+\lambda)}{a_1\cdots a_i}. \end{aligned}$$

If (1.10) is true, then we will have that

$$g(0)>0, \qquad g(1)<0. \tag{4.20}$$

Since $f(\lambda)\to+\infty$ as $\lambda\to+\infty$, we get the result.

(b) $\lambda_4>0$. In this case let $\tilde{a}_i=a_i$, $\tilde{b}_i=b_i+\lambda_4$. Then

$$\begin{aligned} \tilde{f}(\lambda)&=\lambda+\sum_{i=1}^{\infty}(\lambda-i)\frac{(\tilde{b}_0+\lambda)\cdots(\tilde{b}_{i-1}+\lambda)}{\tilde{a}_1\cdots\tilde{a}_i}\\ &=f(\lambda+\lambda_4)-\lambda_4\left(1+\sum_{i=1}^{\infty}\frac{(b_0+\lambda+\lambda_4)\cdots(b_{i-1}+\lambda+\lambda_4)}{a_1\cdots a_i}\right). \end{aligned} \tag{4.21}$$

Let $0<\lambda'<\lambda''$, 0, λ', λ'' be the first three nonnegative solutions of $f(\lambda)=0$.

By (a), we have $0<\lambda'<1, f'(0)=1-1/(1+\lambda_3)>0$. This is then is a $\lambda^*\in(0,\lambda')\ni$ $f(\lambda^*)=\sup_{\lambda\in(0,\lambda')}\{f(\lambda)\}$.

Take T such that

$$0, \lambda', \lambda''\in[0, T), \tag{4.22}$$

$$0<M_T+\sup_{\substack{\lambda\in[0,T)\\ \lambda_4<1}}\left\{1+\sum_{i=1}^{\infty}\frac{(b_0+\lambda+\lambda_4)\cdots(b_{i-1}+\lambda+\lambda_4)}{a_1\cdots a_i}\right\}<\infty. \tag{4.23}$$

Now define

$$C(\lambda_1,\lambda_2,\lambda_3)=\min\left\{\lambda^*, \frac{f(\lambda^*)}{M_T}\right\}>0.$$

For $\lambda_4\in[0, C(\lambda_1,\lambda_2,\lambda_3))$, we have

$$\tilde{f}(\lambda^4-\lambda_4)>0, \quad \tilde{f}(0)<0, \quad \tilde{f}(\lambda''-\lambda_4)<0, \tag{4.24}$$

which implies the result. □

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